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The effect of axial residual stress on the mechanical behaviour of composites

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The effect of axial residual stress on the properties of continuous fibre composites is calculated. It is emphasized that composite strength should be considered as a more complex phenomenon than usually done so by merely considering ultimate strength. It is shown that the information contained in $\sigma(\epsilon)$-curves if plotted in a more suitable form as hysteresis and set* versus total strain, taking into account residual stress, may be a useful means for characterizing the composite and for detecting deviations from perfect structure. Conclusions are derived as to how composite performance may be improved. The results are substantiated by metallic composite data.

1. Introduction

Growing interest in fibrous composites has given rise to a considerable number of publications dealing with mechanical strength of these materials. Favourite topics of these papers are such complex phenomena as crack propagation, delamination, fatigue, creep and their dependence on several variables, some of which are of a complex nature themselves, as thermomechanical treatment, structure of interface layers, arrangement of fibres, and statistics of fibre strength. The reasoning is often kept to a microscopic level, including statements and guesses about dislocation structure in order to explain the outcome of a special experiment. On the other hand, some properties of the composite are based on very simple concepts such as the widely used rule of mixtures. It is somewhat surprising that the simple kind of reasoning in composite problems seems to have never been thoroughly worked out. The aim of this paper is to show an obvious method of calculating certain mechanical composite properties from those of the components as a first approximation. In order to clarify the relations between component and composite properties, all secondary phenomena are neglected. Their effect should be considered later, once the consequences of the simplest possible model have been worked out. We feel this approach may lead to a better understanding of composite behaviour than can be achieved by more sophisticated models which are tailored for special systems.

The subject of this paper is the stress–strain behaviour and related phenomena such as mechanical hysteresis and setting of the material, and their dependence on residual stress. The fatigue of composites will be dealt with in a similar manner in a forthcoming paper.

Our results allow simpler explanations of some observed phenomena than those given in literature. Furthermore, they enable us to question some widespread beliefs, and even to reject some views which have been held to be true hitherto. It is shown how to derive more useful information from $\sigma(\epsilon)$ curves than could be obtained by conventional methods. Among other results, *in situ* $\sigma(\epsilon)$-curves of the matrix are obtained, avoiding the errors usually brought in by calculating the curves via the rule of mixtures.

2. Stress–strain behaviour

2.1. General remarks

The subject of our investigation is continuous fibre composites, especially their response to load in the direction of fibres. As pointed out above, the reasoning is to proceed in terms of the simplest

* Strain remaining after loading.

possible model. We are well aware that the term "simplest possible" is not uniquely defined or self-evident, but we think there will be little dispute about the proposal to ascribe that attribute to a model with the following characteristics:

The model
(1) the components behave ideally elastic–plastic, i.e. without any strain hardening,
(2) the components behave in the composite in the same way as they do separately.

While this simple model provides qualitative understanding rather than quantitative statements about real composites, it describes metallic composites, such as steel wire/aluminium, remarkably well.

Symbols and subscripts are chosen so that their meaning is easy to keep in mind and the danger of confusion with notations of other authors is reduced:

v: volume fraction of fibres
σ, ϵ, E: axial stress, strain, and Young's modulus of the composite, or of matrix or fibre as defined by subscripts M and F respectively
Subscripts 0: at zero stress of the composite
00: at zero stress and zero strain of the composite
σ_0, σ_{00}: residual composite stress (defined by Equation 3)
Subscripts y: yield
y1: yield of only one component of the composite
yh: yield of one component resulting in the formation of hysteresis loops
y2: yield of the two components of the composite
ϵ_s: composite strain remaining after unloading (set)
ϵ_{sc}: characteristic value of the set
η_{el}: stored elastic energy per volume
η: work of hysteresis per volume per cycle

The model is illustrated in Fig. 1. The fact that all reasoning will be for ductile components does not mean that this approach cannot be used to describe brittle ones. Results for composites with one or two brittle components may easily be derived from those of ductile composites on the level of our simple approach.

If the two components deform elastically with changing load of the composite, Young's modulus

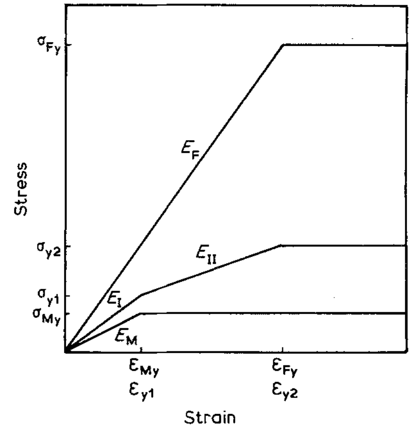

Figure 1 Graphical representation of the assumptions of our model.

of the composite is given by the well-known rule of mixtures

$$E_{\mathrm{I}} = vE_{\mathrm{F}} + (1-v)E_{\mathrm{M}} \quad (1)$$

provided that Poisson's ratio of fibre and matrix coincide. Otherwise some small deviations from Equation 1 should be expected [1], which are neglected in our approach. As soon as one component begins to yield, $d\sigma/d\epsilon$ drops to a value which is usually called the secondary modulus E_{II}. Since we neglect strain hardening as well as lateral constraint, E_{II} is determined only by that component which has not yielded so far:

$$\begin{aligned} E_{\mathrm{II}} &= vE_{\mathrm{F}} \quad \text{if the matrix yields first} \\ E_{\mathrm{II}} &= (1-v)E_{\mathrm{M}} \quad \text{if the fibre yields first.} \end{aligned} \quad (2)$$

In a quantitative description it would read "if $\epsilon_{My} - \epsilon_{M00} \lesseqgtr \epsilon_{Fy} - \epsilon_{F00}$", as will be discussed in connection with Equation 7, but for the sake of convenience we prefer the verbal form.

The ratio σ/ϵ, which is sometimes called the secant modulus, is given no special attention here, since it seems to be hardly justified and misleading, too, to conceive this quantity as a modulus.

2.2. Deduction of $\sigma(\epsilon)$ in the presence of axial residual stress

The presence of residual stress must be anticipated in every composite. In metallic systems it originates from the difference in thermal contraction after preparation of the samples as well as from cold working. We disregard residual forces other than axial, because their effect on the axial mech-

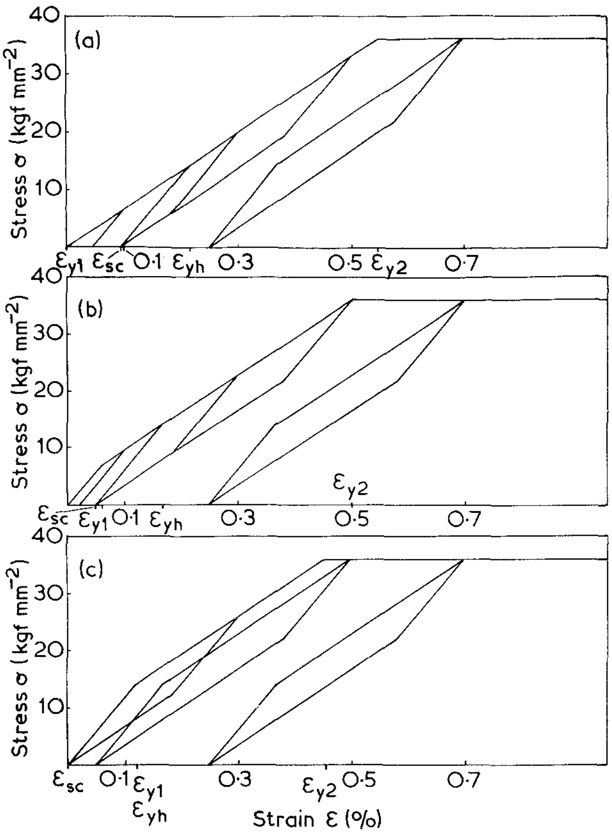

Figure 2 Loading–unloading curves of a model composite with E_F, E_M, ϵ_{Fy}, ϵ_{My}, and ν chosen so as to represent a stainless steel/aluminium composite of fibre volume fraction $\nu = 0.3$: (a) Matrix prestressed at tension, (b) no initial residual stress, (c) fibres prestressed at tension. Note that each of the three plots corresponds to the same loading program: loading up to $\epsilon = 0.1$, 0.3, 0.5, 0.7 and $\geqslant 1.5\%$ with unloading after each stage. The qualitative differences are due to residual stress only.

anical properties can be expected to be small. Since the residual forces of reinforcement and matrix cancel necessarily, only one number is needed to characterize the state of residual stress, which is called σ_0. This quantity may be defined by

$$v\sigma_{F0} = -(1-v)\sigma_{M0} \equiv \sigma_0 \qquad (3)$$

where σ_{F0} and σ_{M0} are the actual stresses in fibre or matrix, respectively, in the absence of external load, i.e. $\sigma = 0$. Thus at $\epsilon = 0$ the components are strained with

$$\epsilon_{M00} = \sigma_{M00}/E_M, \quad \epsilon_{F00} = \sigma_{F00}/E_F \quad (4)$$

(imagine matrix under compression, for instance, i.e. $\epsilon_{M00} < 0$). Henceforth we will reason in terms of initial residual stress at zero composite strain, which is indicated by the subscript 00. If the composite is loaded with increasing tensile load $\sigma > 0$, the matrix compression is first released, and the resulting strains of the components are

$$\epsilon_M = \epsilon_{M00} + \epsilon \qquad (5a)$$

$$\epsilon_F = \epsilon_{F00} + \epsilon \qquad (5b)$$

These relations reflect the obvious fact that in continuous fibre composites all axial strain changes due to axial load are exactly the same for fibre, matrix, and the whole composite, disregarding the end regions of the specimen.

As soon as one of the components begins to yield, the composite changes its behaviour. The composite strain at which this qualitative change occurs is called here ϵ_{y1}. Equation 5 leads to the relation

$$\begin{aligned} \epsilon_{My} &= \epsilon_{M00} + \epsilon_{y1} \text{ if the matrix yields first} \\ \epsilon_{Fy} &= \epsilon_{F00} + \epsilon_{y1} \text{ if the fibre yields first.} \end{aligned} \qquad (6a)$$

For strains not too far above ϵ_{y1}, the $\sigma(\epsilon)$-plot of the composite looks like that of a strain-hardening material (Fig. 2b). If the matrix yields first, as in Fig. 2, the composite behaves in the following way: in the process of unloading, when the composite contracts as a whole, the matrix strain decreases and turns from tension to compression. We assume that the matrix yield strain for tension and compression are the same, which is a good approximation for ductile materials. The elastic contraction of the composite during unloading, σ/E_1, cannot then exceed $2\epsilon_{My}$. All further contraction must be accompanied by matrix flow, implying a slope E_{II} of that part of the downward $\sigma(\epsilon)$-curve (Fig. 2). As may be easily seen, such a behaviour gives rise to hysteresis loops as a result of repeated loading. The strain at which hysteresis loops emerge with our model composite may be called ϵ_{yh}. Provided such a strain exists, its value follows from the above considerations via

$$\begin{aligned} \sigma_{yh}/E_I &= 2\epsilon_{My} \\ \sigma_{yh} &= \epsilon_{y1}E_I + (\epsilon_{hy} - \epsilon_{y1})E_{II}. \end{aligned} \quad (6b)$$

Obviously, a third critical strain must exist, which is related to the yield of the remaining component. Analogous to the deduction of Equation 6a we obtain from Equation 5

$$\begin{aligned} \epsilon_{Fy} &= \epsilon_{F00} + \epsilon_{y2} \text{ if the matrix yields first} \\ \epsilon_{My} &= \epsilon_{M00} + \epsilon_{y2} \text{ if the fibre yields first.} \end{aligned} \quad (6c)$$

Some composites exist where their reinforcement is not strong enough to cause the matrix to yield under compression when the composite is unloaded. If the reinforcement yields first, which may occur even with strong-fibre composites, no hysteresis due to matrix flow can occur. On the other hand, the hysteresis loop of the composite of Fig. 2 decreases, the closer ϵ_{yh} and ϵ_{y2} are brought together. Thus we have a condition for the occurrence of hysteresis due to matrix flow: ϵ_{yh} from Equation 6b must be lower than ϵ_{y2} from (Equation 6c, first line).

For the sake of completeness, the possibility of hysteresis due to fibre flow should be mentioned, i.e. the existence of another value ϵ_{yh}. The corresponding formula can be obtained from the former by interchanging fibre and matrix, i.e. $F \leftrightarrow M$ and $v \leftrightarrow 1 - v$.

Having outlined the basic ideas, we may proceed to formulate explicit expressions for ϵ_{y1}, ϵ_{yh}, and ϵ_{y2} instead of those implicit in Equations 6a to c.

$$\epsilon_{y1} = \begin{cases} \epsilon_{My} - \epsilon_{M00} \\ \epsilon_{Fy} - \epsilon_{F00} \end{cases} \text{whichever is the smaller.} \quad (7a)$$

$$\epsilon_{yh} = \epsilon_{My}\left(\frac{E_I}{E_{II}} + 1\right) - \epsilon_{F00}, \qquad \text{if } \epsilon_{My}\left(\frac{E_I}{E_{II}} + 1\right) < \epsilon_{Fy}$$

$$\epsilon_{yh} = \epsilon_{Fy}\left(\frac{E_I}{E_{II}} + 1\right) - \epsilon_{M00} \quad (7b)$$
$$\text{if } \epsilon_{Fy}\left(\frac{E_I}{E_{II}} + 1\right) < \epsilon_{My}$$

otherwise ϵ_{yh} is non-existent.

$$\epsilon_{y2} = \begin{cases} \epsilon_{Fy} - \epsilon_{F00} \\ \epsilon_{My} - \epsilon_{M00} \end{cases} \text{whichever is the greater.} \quad (7c)$$

Note that ϵ_{M00} and ϵ_{F00} are connected by Equations 3 and 4. The two alternatives in Equations 7a and c as on former occasions, refer to the two possible cases of one component yielding before the other. The internal stress can be chosen so that, at least theoretically, the two alternatives coincide:

$$\begin{aligned} &\epsilon_{My} - \epsilon_{M00} = \epsilon_{Fy} - \epsilon_{F00} \\ &\text{if } \sigma_{00} = (\epsilon_{Fy} - \epsilon_{My})\frac{vE_F(1-v)E_M}{vE_F + (1-v)E_M} \equiv \sigma_{00}^* \end{aligned} \quad (8)$$

The composite stresses belonging to the strains ϵ_{y1}, ϵ_{yh}, ϵ_{y2} are called σ_{y1}, σ_{yh}, σ_{y2}. Using Equation 6b and the relations

$$\begin{aligned} \sigma_{y1} &= \epsilon_{y1}E_I \\ \sigma_{y2} &= \sigma_{y1} + (\epsilon_{y2} - \epsilon_{y1})E_{II} \end{aligned} \quad (9)$$

they can be obtained easily as functions of the parameters ϵ_{My}, ϵ_{Fy}, E_M, E_F, v, and σ_{00}:

$$\sigma_{y1} = [(1-v)\sigma_{My} + \sigma_{00}]\left(\frac{v}{1-v}\frac{E_F}{E_M} + 1\right) \qquad \text{if } \sigma_{00} < \sigma_{00}^*$$

$$(10a)$$

$$\sigma_{y1} = (v\sigma_{Fy} - \sigma_{00})\left(\frac{1-v}{v}\frac{E_m}{E_F} + 1\right) \qquad \text{if } \sigma_{00} > \sigma_{00}^*.$$

$$\sigma_{yh} = 2\sigma_{My}\left(v\frac{E_F}{E_M} + 1 - v\right) \qquad \text{if } (1-v)\sigma_{My} < \sigma_{00}^*$$

$$\sigma_{yh} = \text{non-existent} \quad (10b)$$
$$\text{if } -v\sigma_{Fy} < \sigma_{00}^* < (1-v)\sigma_{My}$$

$$\sigma_{yh} = 2\sigma_{Fy}\left((1-v)\frac{E_M}{E_F} + v\right) \qquad \text{if } \sigma_{00}^* < -v\sigma_{Fy}$$

$$\sigma_{y2} = v\sigma_{Fy} + (1-v)\sigma_{My}. \quad (10c)$$

Among these quantities, which characterize different aspects of composite strength, only σ_{y1}, which might be the one of most technical interest, depends on residual stress. The consequences of Equation 10 will be discussed later.

With the characteristic values known, it is easy, of course, to construct $\sigma(\epsilon)$-curves of any given loading regime in graphic or analytical form.

2.3. Plastic deformation

The set of the material, i.e. the deformation which remains after unloading, shows a peculiar dependence on maximum strain ϵ which the composite has been subjected to. In the elastic region, i.e. up to ϵ_{y1}, the set ϵ_s is zero. With increasing ϵ it grows like that of a strain-hardening material. When ϵ has exceeded ϵ_{yh}, the pull in the fibres has become so strong that towards the end of the unloading process, the matrix yields under the compressive load exerted on it by the fibres. Thus the composite will be pulled back to a certain set ϵ_{sc} (Fig. 2). Any further increase of ϵ brings the composite again back to ϵ_{sc} after unloading, provided ϵ remains below ϵ_{y2}. Otherwise ϵ_s increases further, because both components are then yielding.

From simple geometrical reasoning using Fig. 2, we obtain

$$\begin{aligned}
\epsilon_s &= 0 \quad \text{if} \quad 0 < \epsilon < \epsilon_{y1} \\
\epsilon_s &= (\epsilon - \epsilon_{y1}) \frac{E_I - E_{II}}{E_I} \quad \text{if} \quad \epsilon_{y1} < \epsilon < \epsilon_{yh} \\
\epsilon_s &= (\epsilon_{yh} - \epsilon_{y1}) \frac{E_I - E_{II}}{E_I} \quad \text{if} \quad \epsilon_{yh} < \epsilon < \epsilon_{y2} \\
\epsilon_s &= (\epsilon_{yh} - \epsilon_{y1}) \frac{E_I - E_{II}}{E_I} + \epsilon - \epsilon_{y2} \quad \text{if} \quad \epsilon_{y2} < \epsilon.
\end{aligned} \tag{11a}$$

The constant value which ϵ_s assumes for $\epsilon_{yh} < \epsilon < \epsilon_{y2}$ is the number ϵ_{sc} mentioned above. If ϵ_{yh} does not exist, Equation 11a is simplified to

$$\epsilon_s = \begin{cases} 0 \\ (\epsilon - \epsilon_{y1}) \dfrac{E_I - E_{II}}{E_{II}} \\ (\epsilon_{y2} - \epsilon_{y1}) \dfrac{E_I - E_{II}}{E_{II}} + \epsilon - \epsilon_{y2} \end{cases} \tag{11b}$$

In this case no constant ϵ_{sc} exists.

$$\epsilon_{sc} = \begin{cases} \text{non-existent if } \epsilon_{yh} \text{ does not exist} \\ (\epsilon_{yh} - \epsilon_{y1}) \dfrac{E_I - E_{II}}{E_I} \quad \text{if } \epsilon_{yh} \text{ does exist.} \end{cases} \tag{12}$$

Since ϵ_{y1} depends on residual stress, ϵ_s and ϵ_{sc} depend on it, too. The result of Equation 11 is discussed in Section 3.2.

2.4. Stored energy and hysteresis

Since the components of our idealized composite are to exhibit no hysteresis, in this section we shall only deal with that kind of hysteresis which is an intrinsic property of some types of fibrous composites, as is illustrate by the loops of Fig. 2. The area below the unloading branch of the $\sigma(\epsilon)$-curve represents the stored elastic energy per volume, η_{el}. Elementary calculations lead to the formulae

$$\begin{aligned}
\eta_{el} &= (\epsilon - \epsilon_{yh} + 2\epsilon_{My})^2 E_{II}/2 + 2\epsilon_{My}^2 (E_I - E_{II}) \\
&\qquad \text{if } \epsilon_{yh} < \epsilon < \epsilon_{y2} \\
&= \left(\epsilon + \frac{\sigma_{00}}{vE_F} - \epsilon_{My} \frac{(1-v)E_M}{vE_F}\right)^2 vE_F/2 \\
&\qquad + 2\epsilon_{My}^2 (1-v) E_M.
\end{aligned} \tag{13}$$

(For the sake of simplicity, the symbol ϵ has been assigned to the maximum strain of the cycle as well as to the strain variable.) In the region $\epsilon_{y2} < \epsilon$, η_{el} is constant at the value $\eta_{el}(\epsilon_{y2})$ according to Equation 11.

The stored energy being known, the energy loss due to hysteresis, which is the area of the loop, can be obtained by subtracting $2\eta_{el}$ from the area of the rectangle in which the loop is situated, with the sides $\epsilon - \epsilon_{sc}$ and $\sigma(\epsilon)$: $\eta = (\epsilon - \epsilon_{sc})\sigma - 2\eta_{el}$, provided that hysteresis does exist.

$$\eta = \begin{cases} 0 \\ \quad \text{if } \epsilon < \epsilon_{yh} \text{ or } \epsilon_{yh} \text{ does not exist} \\ 2(\epsilon - \epsilon_{yh})\epsilon_{My}(E_I - E_{II}) \\ \quad \text{if } \epsilon_{yh} < \epsilon < \epsilon_{y2} \text{ and } \epsilon_{yh} \text{ does exist} \\ 2(\epsilon_{y2} - \epsilon_{yh})\epsilon_{My}(E_I - E_{II}) \\ \quad \text{if } \epsilon_{y2} < \epsilon \text{ and } \epsilon_{yh} \text{ does exist.} \end{cases} \tag{14}$$

As can be seen from Equations 14 and 13, the ratio of hysteretic loss to stored elastic energy, η/η_{el}, is a non-linear function of ϵ. This will be discussed in detail in Section 3.2.

3. Discussion of the results

3.1. Results related to strength

Our results are aimed at both the understanding of composite behaviour in principle and the providing

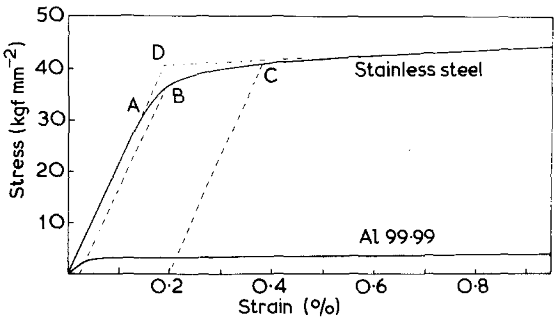

Figure 3 Stress–strain curves of stainless steel and Al (99.99% pure), which show approximately the behaviour required for our model composite of Fig. 1

of approximate data of any special fibrous composite system. For the latter purpose the quantities of our idealized system must be related to those of the real ones. This may depend on the property in question as well as on the required precision. If one is interested in the elastic limit only, for σ_{My} one would choose the elastic limit of the matrix instead of matrix yield stress. Depending on the problem, for σ_{Fy} one may choose one of the values at A, B, C or D in Fig. 3 or even the ultimate stress.

According to this moderate arbitrariness, σ_{y1} may denote either the elastic limit or the yield stress of the composite, and σ_{y2} may be the ultimate stress or merely that stress at which the composite as a whole begins to flow. These hints for proper use of Equations 7 and 8 will not concern us further.

As a widespread phenomenon in composite research, efforts continue to produce reliable strength values close to the rule-of-mixtures data. As Cunningham and Alexander [2] stated, "the effort. . . can be summarized as stressing the elimination of filament breakage, the minimization of filament degradation, and the development of an adequate filament- matrix bond". These efforts may be necessary though surely not sufficient to create an "advanced family of superior materials": high ultimate strength may be useful as a means of preventing catastrophic breakdown resulting from overload; σ_{y1} according to Equation 10a, however, seems to be a much more important composite property. As a consequence, efforts should be made to maximize σ_{y1} instead of ultimate stress.

In the majority of conceivable applications, the load must not exceed σ_{y1}. At stresses below σ_{y1} all phenomena promoting early damage are absent, at least in the first approximation, which our model intends to provide. Therefore, the effect of residual stress on σ_{y1} should be of great interest.

Since σ_0 cannot exceed certain limits due to matrix flow or fibre failure,

$$|\sigma_0| \leqslant \begin{cases} v\sigma_{Fy} \\ (1-v)\sigma_{My} \end{cases} \text{whichever is the smaller,} \quad (15)$$

the region that may be swept over by σ_{y1} of the same composite is also limited. Let us consider, for instance, the case of aluminium reinforced with steel wires ($E_F/E_M = 3$, $\epsilon_{Fy}/\epsilon_{My} = 10$). At reasonable volume fractions, the inequality $v\sigma_{Fy} > (1-v)\sigma_{My}$ holds, which reduces Condition 13 to $|\sigma_0| \leqslant (1-v)\sigma_{My}$. Since in this special case $\sigma_{00}^* = 27\sigma_{My}\, v(1-v)/(1+2v)$ according to Equation 8, even the largest possible value of σ_0, and hence σ_{00}, is smaller than σ_{00}^*, unless v is very small. Thus we are dealing with a composite whose matrix will yield under tension before the fibres do so. Consequently, the first formula of σ_{y1} in Equation 10 is applicable to our material. Inserting the largest possible values which σ_{00} can assume, and $\sigma_{00} = 0$ for comparison, we obtain

$$\sigma_{y1} = \begin{cases} 0 \\ \epsilon_{My}E_I \\ 2\epsilon_{My}E_I \end{cases} \quad \text{if} \quad \sigma_{00} = \begin{cases} -(1-v)\sigma_{My} \\ 0 \\ (1-v)\sigma_{My} \end{cases}$$

provided that $v > 0.04$, $E_F/E_M = 3$, $\epsilon_{Fy}/\epsilon_{My} = 10$. This means that σ_{y1}, which amounts to $\epsilon_{My}E_I$ in the absence of residual stress, may vary between zero and twice of that value.

Since σ_{00} of a given composite material is generally unknown, it would be desirable to know in advance the limits for σ_{y1}. From Equation 7 it follows that it is possible in every case to prestress the composite so that $\epsilon_{y1} = 0$ and hence $\sigma_{y1} = 0$. Calculating the upper limit, we must distinguish three types of composites, which will be explained with the help of a graphical representation (Fig. 4).

Composites with $\epsilon_{My} = \epsilon_{Fy}$ have values $\sigma_{y1} = v\sigma_{Fy} + (1-v)\sigma_{My} = \sigma_{y2}$ in the absence of residual stress, $\sigma_{00} = 0$. This is, of course, the highest possible value of σ_{y1}. Moving in the $(\sigma_{My}, \sigma_{Fy})$ plane of Fig. 4 from the line $\epsilon_{My} = \epsilon_{Fy}$ to the right, where $\epsilon_{Fy} > \epsilon_{My}$, we find composites with an increasing difference between σ_{y1} and σ_{y2}, in the absence of residual stress. By prestressing in the right way we can still get σ_{y1}, as high as σ_{y2}:

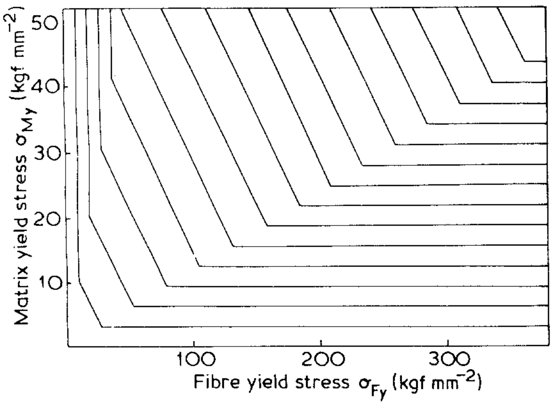

Figure 4 Composite yield stress σ_{y_1} (with only one component yielding) according to Equation 10a for various combinations of component yield stress σ_{Fy} and σ_{My}, in the presence of optimum residual stress according to Equation 16b; $E_F/E_M = 3$, $v = 0.3$.

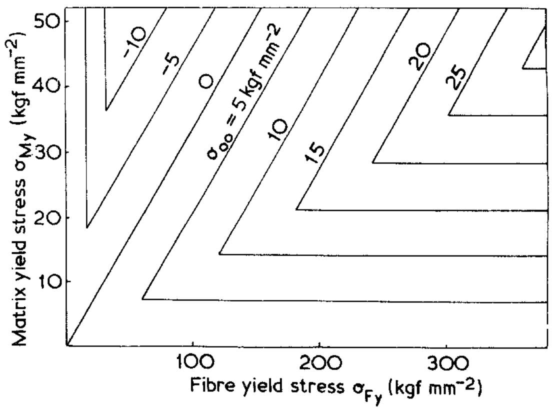

Figure 5 Residual stress σ_{00} required for maximizing σ_{y_1} according to Equation 16b, i.e. the optimum residual stress of Fig. 4.

This result is obtained by putting $\sigma_{00} = \sigma_{00}^*$ in Equation 10. Proceeding further to the right in Fig. 4, we find composites whose matrix is no longer able to withstand the residual stress required for σ_{y1} to equal σ_{y2}. The boundary beyond which these composites are placed with a maximum residual matrix stress of $\sigma_{M00} = -\sigma_{My}$ (that is $\sigma_{00} = (1-v)\sigma_{My}$) is given by the straight line $(1-v)\sigma_{My} = \sigma_{00}^*$. Such composites sustain elastic deformation up to $2\epsilon_{My}$, corresponding to a stress of $2\epsilon_{My}E_I$, which turns out to be σ_{yh}, of Equation 10. In the half plane $\epsilon_{My} > \epsilon_{Fy}$, the situation is analogous. There all results follow from those for $\epsilon_{My} < \epsilon_{Fy}$ by interchanging F $\leftrightarrow$ M and $(1-v) \leftrightarrow v$ in the formulae. To sum up,

$$\text{if} -v\sigma_{Fy} < \sigma_{00}^* < (1-v)\sigma_{My}$$

$$\text{then } 0 < \sigma_{y1} < \sigma_{y2} \tag{16a}$$

$$\text{otherwise } 0 < \sigma_{y1} < \sigma_{yh}.$$

The residual stress required for σ_{y1} to coincide with the upper limit in Equation 16a, i.e. "optimum residual stress", is consequently given by the following expression:

$$\begin{aligned} \sigma_{00} &= (1-v)\sigma_{My} && \text{if } (1-v)\sigma_{My} < \sigma_{00}^* \\ \sigma_{00} &= -v\sigma_{Fy} && \text{if } -v\sigma_{Fy} > \sigma_{00}^* \\ \sigma_{00} &= \sigma_{00}^* && \text{otherwise} \end{aligned} \tag{16b}$$

(see Fig. 5). Thus σ_{00}^* of Equation 8 is the optimum residual stress of a certain class of composites.

The formulae 16 and the example given above emphasize the importance of residual stress. The question now is what magnitude those stresses are expected to be in reality. One possible source of residual stress is different thermal contraction after hot pressing. A crude approximation can be obtained by setting up the balance of forces in fibre direction, neglecting all that may happen perpendicular to the fibres:

$$\sigma_{00} = \frac{(E_I - E_{II})E_{II}}{E_I}(\alpha_F - \alpha_M)(T_2 - T_1). \tag{17}$$

Hence $\sigma_{00} < 0$, if the composite is stress-free at the higher temperature T_2, and the coefficients of thermal expansion obey the relation $\alpha_F < \alpha_M$. Unfortunately, this situation is met in most metallic composites produced by hot-pressing or subjected to thermomechanical treatment. Therefore, as-

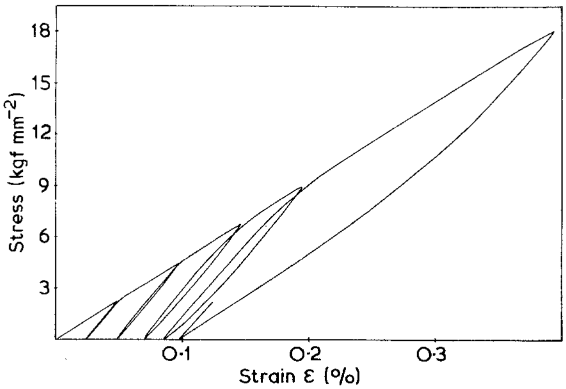

Figure 6 Experimental $\sigma(\epsilon)$ curve of a steel wire/Al (99.99% pure) composite with $v = 0.3$, where $\sigma_{y_1} = 0$ because of residual stress due to hot pressing.

pressed samples of steel wire /Al-composites may show $\sigma_{y1} = 0$ as in Fig. 6. Such a performance, of course, is highly undesirable. This $\sigma(\epsilon)$-plot of Fig. 6, however, as well as that of Fig. 2, shows a simple means to overcome the difficulty: after stretching there are other residual stresses which are of a more desirable nature. If the total strain reached ϵ_{yh} (or ϵ_{y2} in those cases where ϵ_{yh} does not exist) the useful stress assumes its largest possible value. Thus ϵ_{yh} (or ϵ_{y2}) is a quantity important for cold-working. Compared with the degree of deformation usually met with cold-working, ϵ_{yh} is a very small strain. This means that samples which underwent final cold-working are certainly prestressed in the desired way. Since, in general, very small degrees of deformation, (less than 1%) are sufficient to generate the desired residual stress, it should be possible to prestress even composites with brittle fibres by this method, if special care is taken to obtain definite strain.

Fig. 7 shows the $\sigma(\epsilon)$-curve of an optimally prestressed composite at repeated loading.

In order to emphasize the flexibility of our scheme, we will demonstrate how information on other types of systems can be derived almost immediately from our results. If one wants to take into account fibre or matrix fracture in addition to, or instead of, yield, one may start from Equation 5: that composite strain which causes fracture of one component is called ϵ_{f1}. If the filament breaks first, then $\epsilon_{Ff} = \epsilon_{F00} + \epsilon_{f1}$. If ϵ_{f1} is known, σ_{f1} can be derived. By comparison with the strength of the remaining component, it can be decided whether this is the ultimate strength or not. Since no additional ideas are required to derive the corresponding formulae, we will confine ourselves to these few remarks.

Substituting y (yield) by f (fracture) in Equation 10a, we obtain the formula for σ_{f1} of brittle composites. The only thing remaining to be done is to modify Equation 15, since tensile and compressive strength of brittle materials differ considerably.

We will not undertake the tedious task of investigating all types of composites which are generated by permutation of the series $\epsilon_{My} < \epsilon_{Fy} < \epsilon_{Mf} < \epsilon_{Ff}$, the number of which could still be increased by allowing the components to exhibit different behaviour under compression and tension. The properties of any system of interest may, of course, be worked out easily by proceeding along the lines given above.

A recent paper by Sorsorov *et al., Composites* 7 (1976) 17, seems to contain an extension of our ideas to a model composite with a work-hardening matrix. Ustinov, one of the authors, became acquainted with our work at a joint working session in Moscow in 1974.

3.2. Results related to yield of one component

Composite properties caused by yield are of interest for fabricational and quality controlling purposes rather than representing useful properties of the final product. There are possible exceptions: the huge hysteresis loop might be useful for damping dangerous oscillations which otherwise would cause catastrophic events.

The hysteretic loss does not change monotonically with v since it must vanish at $v = 0$ as well as at $v = 1$ and, therefore, must exhibit a maxi-

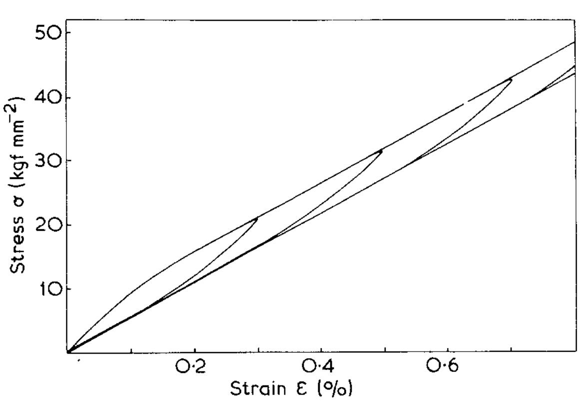

Figure 7 Experimental $\sigma(\epsilon)$ curve of a high-strength steel wire/Al (99.99% pure) composite with $v = 0.3$, where the set $\epsilon_s = 0$ because of residual stress due to stretching after hot pressing.

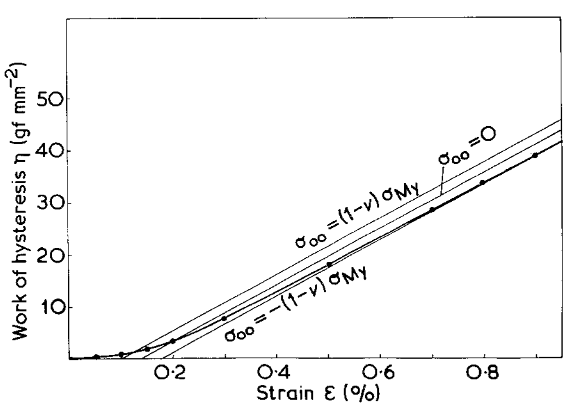

Figure 8 Hysteretic loss of high-strength steel wire/Al (99.5% pure) composite with $v = 0.3$, superimposed on a family of model curves according to Equation 14.

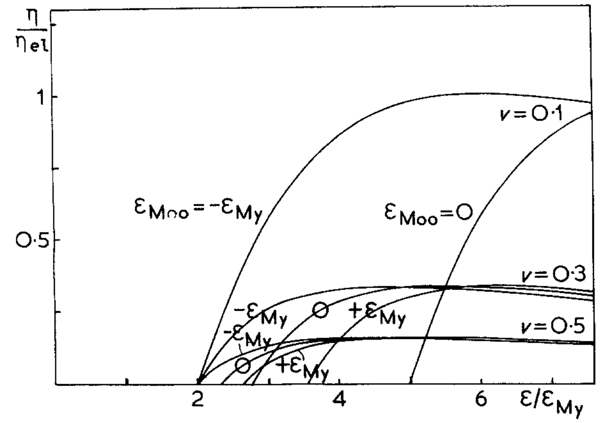

Figure 9 Ratio of hysteretic loss to stored elastic energy according to Equations 13 and 14; $E_F/E_M = 3$, $v = 0.1$, 0.3, and 0.5.

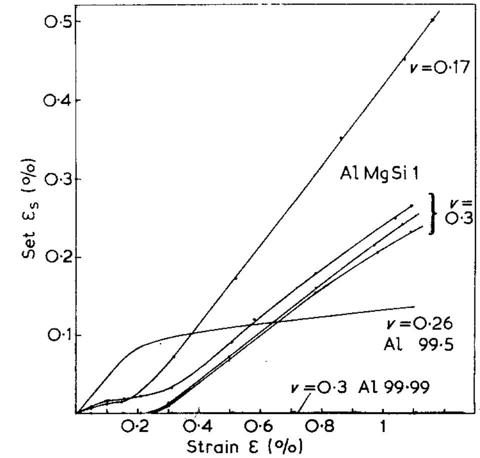

Figure 10 The setting of some high-strength steel wire/aluminium composites after straining to ϵ.

mum at some value of v in between: the maximum of $\eta(\epsilon_{y2})$, for instance, is reached at

$$v = \sqrt{\left(\frac{\epsilon_{My}E_M}{\epsilon_{My}E_M + (\epsilon_{Fy} - 2\epsilon_{My})E_F}\right)}.$$

In general, real composites behave in this respect nearly like ideal ones, but at strains not too close to ϵ_{y2}.

Fig. 8 shows the work of hysteresis plotted versus the maximum strain of the cycle for our idealized composite according to Equation 14 as well as an experimental curve from a steel wire/Al-composite. Note that the experimental curve tends rapidly toward zero at small ϵ. Badly prepared samples show much larger hysteresis at small ϵ. Thus hysteresis measurement is a good means of determining the quality of semi-finished composite products.

Hysteresis data are often directly obtained as η/η_{el}; this rather unwieldy function derived from Equations 13 and 14 is illustrated in Fig. 9. It can be clearly seen that residual stresses shift the curves along the ϵ-axis.

Like hysteresis, the set $\epsilon_s(\epsilon)$ is generally not desired, but is able to provide information on deviations from the ideal structure of the composite. Furthermore, the residual stress σ_{00}, the detection of which by direct means requires some effort [3], can easily be derived from the plots of $\epsilon_s(\epsilon)$ or $\eta(\epsilon)$. For this purpose the experimental curve must be compared with the family of curves arising from Equations 11 or 14, respectively, by variation of σ_{00} (Fig. 8). Such a procedure would also work well with Fig. 10; however, we have not done so, in order to keep the figure as uncomplicated as possible in order to point out other features: the piecewise linear shape of the curves $\epsilon_s(\epsilon)$ of our model composites, which is stated in Equation 11, is observed very accurately with some real composites. This serves as further evidence for the usefulness of our simple model. On closer consideration, significant deviations from the ideal behaviour are revealed. Two of the four alloy matrix curves of Fig. 10, which are expected to be straight lines right down to the ϵ-axis, deviate shortly before reaching it, which indicates departure from perfect structure.

The optimally prestressed sample of Fig. 7 shows no setting at all: since ϵ_{y1} and ϵ_{yh} coincide in this case, ϵ_s equals zero up to $\epsilon = \epsilon_{y2}$ according to Equation 11a, which means that our curve coincides with the abscissa axis. Another sample with an Al(99.5% pure) matrix, behaves in a completely different way. This can be explained by the reversed sign of residual stress compared with the former (curve begins to rise from $\epsilon = 0$ onward), combined with misalignment (non-zero slope at $\epsilon > \epsilon_{yh}$). More of our results are shown in Fig. 14, which is commented on within the next section.

4. Comments on other work

In addition to our introductory remarks which explained the reasons for our simple approach, further justification might be derived from the fact that we have, by this reasoning, detected some erroneous results and statements in papers of more complex and sophisticated investigations of special systems. In a paper on fatigue of fibre-reinforced aluminium [4], Baker is of the opinion that it would be desirable to avoid large internal stresses, which should be done by using matrices of low yield stress; hence, "it is desirable to have a matrix of high yield stress only when the fibres are relatively weak. . .". However, we feel that based upon our results, metallic composites of high-strength fibres and very weak matrix are of little use. Unlike the situation in composites with a resin matrix, where despite the low strength of the matrix its elastic strain limit is high, metallic systems having a low matrix strength imply a low yield strain because of the small modulus ratio E_F/E_M, resulting in low σ_{y1}. Thus σ_{y1} of composites of pure Al matrix with $\sigma_{My} = 4.5\,\mathrm{kgf\,mm^{-2}}$ *in situ* matrix yield stress, and 30 vol% high-strength fibres, amounts to no more than $\sigma_{y1} = 2\epsilon_{My}E_I = 14\,\mathrm{kgf\,mm^{-2}}$, no matter how strong the fibres are. This is roughly the stress at which the number of loading cycles to failure decreases rapidly, whereas the high-strength steel wires, for instance, have been stressed up to as little as 4% of their strength. The reinforcement of metals by high-strength fibres must, therefore, preclude the use of unduly weak matrices, perhaps with the exception of some rather particular applications. Baker suggests that residual stresses may be avoided by using weak matrices which entails eliminating stresses by expelling composite strength. Our results suggest an alternative approach by turning the tendency to build up residual stresses into an advantage, by generating stresses of a useful nature, as discussed in Section 3.1.

Baker's Equation 16 for damping capacity corresponds to our result for the special case of $\sigma_{00} = (1 - v)\sigma_{My}$. The derivation of this formula, however, is not correctly given in [4]: for instance, Baker interprets the term $2(1 - v)\sigma_{My}^2/E_M$, as elastic energy contained in the matrix. However, the matrix is unable to store more than a quarter of that amount of elastic energy. There is also a misprint in Equation 17 of [4].

Varshavsky [5] gives an equation for the damping capacity which agrees with ours in the approximation $\eta_{el} \approx \sigma\epsilon/2$. However, our interpretation is entirely different from that given by Varshavsky. In Fig. 3 of [5], the hysteresis loops can be seen distinctly even at strains as low as $\epsilon = 0.5 \times 10^{-3}$. The aluminium alloy with its yield stress of $\sigma_{My} = 28\,\mathrm{kgf\,mm^{-2}}$ should exhibit no essential deviation from the elastic straight line in its $\sigma(\epsilon)$-plot up to about $\epsilon = 3 \times 10^{-3}$. As the fibres will also behave purely elastically, there should be no hysteresis evident in Varshavsky's Fig. 3. The comparatively large loops observed experimentally must, therefore, be due to causes other than overall matrix yield, i.e. misalignment of fibres or inhomogeneous stress distribution inside the sample. Under such circumstances the tangent modulus at the upper point of the loop is in no way connected with the secondary composite modulus $E_{II} = vE_F$. If all the experimental tangents appear to have a common slope E_{II} (E_2 in Varshavskys notation), a certain ambiguity in the drawing of the tangent, as well as the desire to obtain E_{II}, may account for this. The monotonous $\sigma(\epsilon)$ curves of Fig. 4 of [5] resemble the ideal ones because of their exact linearity, which is inconsistent with the observations mentioned above. The slopes of the curves coincide with the data of Table I of [5]. Thus the monotonous curves were probably not observed directly, but constructed from averaged cyclic data. They cannot, therefore, serve as independent experimental evidence. We feel, therefore, that consequences derived from his experimental results, i.e. the very high *in situ* matrix modulus of 17 400 $\mathrm{kgf\,mm^{-2}}$, the surprisingly strong dependence of matrix properties on fibre content (Fig. 5 and 6 of [5]), and the unreasonable values of the fraction of load carried by the fibres (Fig. 7 of [5]), should be rejected. Varshavsky's Fig. 11 appears to be an adaptation of his Fig. 4 by means of his Equation 17, and thus contains no extra experimental information.

Several other papers report the observation of a considerable increase of *in situ* matrix strength due to the presence of fibres [6–8]. These observations prompted some authors to attempt an explanation of the effect [8–13]. It is worthwhile to reconsider the *in situ* matrix data in the light of our findings. By a reconstruction of matrix stress–strain curves from those of composite and reinforcement (Figs. 2 and 3 of [8]) we obtained a different family of curves (our Fig. 11) than those of Lee and Harris (Fig. 4 of [8]). In Fig. 3 [8] the slope of the 48 μm curve is too steep when compared with the modulus data of Table I [8]. Assuming that the 48 μm data of Fig. 3 [8] had a systematic error of 5%, which seems likely because of the different initial slope, the derived 48 μm matrix curve fits very well into the rest of the family. Up to this point residual stress has not

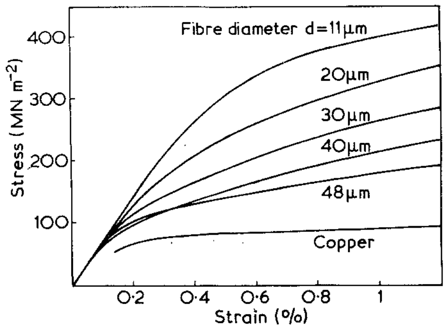

Figure 11 *In situ* matrix stress-strain curves redrawn from the results of Lee and Harris [18] on copper/tungsten composites.

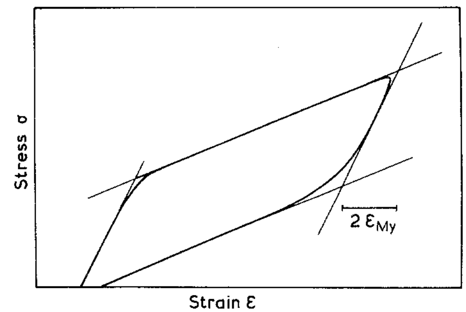

Figure 13 Derivation of *in situ* matrix behaviour from hysteresis loops.

been considered. Fig. 11 has been derived with the presumption that residual stress is absent. In the presence of residual stress the result is somewhat different. The rule of mixtures takes the form

$$\sigma(\epsilon) = v\sigma_F(\epsilon + \epsilon_{F00}) + (1 + v)\sigma_M(\epsilon + \epsilon_{M00}) \quad (18)$$

where ϵ_{F00} and ϵ_{M00} are connected by $0 = v\sigma_F(\epsilon_{F00}) + (1 - v)\sigma_M(\epsilon_{M00})$, which is Equation 18 at $\epsilon = 0$. The effect of residual stress on the *in situ* matrix behaviour is shown in Fig. 12, which has been derived with the aid of Equation 18 from the composite and fibre data of [8]. It may be interpreted in the following way. If the fibres are prestressed at tension but the matrix curve is derived as if there were no residual stress, the resultant curve is too high. For reasons given above, the virgin samples of Lee and Harris can be expected to be prestressed at $\sigma_{00} < 0$, whereas in the course of cyclic testing, σ_0 rises considerably above zero. As a consequence, the monotonic matrix curves were too low, and cyclic ones too high in the calculations of Lee and Harris. Therefore, the whole or part of the difference between Figs. 4 and 5 of [8], may be caused by simply neglecting residual stress.

For systems exhibiting distinct hysteresis, e.g. Fig. 7, the *in situ* $\sigma(\epsilon)$ curve of the matrix can be derived easily: draw the tangents at the points of zero stress and maximum stress at each loop. If the loops show deviations from ideal shape, proceed as indicated in Fig. 13. After having constructed in this way an idealized counterpart of the real loop, one immediately obtaines $2\epsilon_{My}$ and hence σ_{My} as a function of the maximum cyclic strain. Care must be taken that the loops subjected to this procedure are sufficiently large in order to show clearly the slope E_{II}. Fig. 14 shows some curves

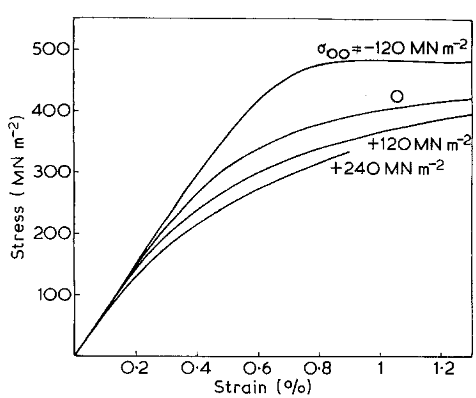

Figure 12 *In situ* matrix stress-strain curves redrawn from the results of Lee and Harris [8] for a fibre diameter $d = 11\,\mu m$, taking into account various amounts of residual stress according to Equation 18.

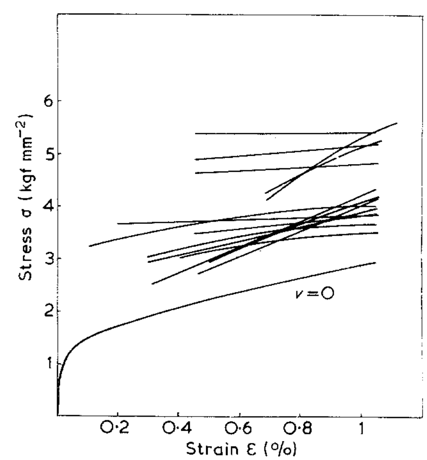

Figure 14 *In situ* matrix stress-strain curves (smoothed) of high-strength steel wire/Al (99.5% pure) composite, derived from hysteresis loops; $v = 0.1$ to 0.3.

derived in this way for an Al matrix. The samples were taken from different series. As a result, the curves tend to group themselves into families with common features. Although detailed investigations have not yet been performed, there is no doubt that these features are related to technological parameters and that they carry information about the microstructure of the composite.

Our method has the advantage of not being affected by residual stress or uncertainties in volume fractions and load carried by the fibres, which make some of the *in situ* matrix curves derived using the rule of mixtures highly questionable. Hancock and Grosskreutz [14], for instance, find $\sigma(\epsilon)$ of the matrix *in situ* to be smaller than that of the compact matrix, and, in addition, they observe small values of σ_{y1} (our notation). However, they do not consider that axial residual stress could be the cause of both observations. Furthermore, the axial component is neglected explicitly because they claim that transverse residual stress is important. According to our investigations, the contrary is true: the axial component of residual stress gives rise to remarkable phenomena, whereas the transverse components may be neglected in a first approximation.

The remarkably high rate of strain hardening of the single crystal copper matrix reported by Kelly and Lilholt [6] is not questioned when considered from our point of view. Neglecting residual stress in the calculation of *in situ* $\sigma(\epsilon)$ may alter the result by a factor of 0 to 2 at the most, which follows from Equations 10a and 15. The relative error due to neglect of residual stress in [6] seems to be very small.

The neat microstrain hysteresis loops of steel wire/Al composite reported by Pinnel and Lawley [15] are not to be confused with the loops considered in this paper, as they are of another origin. Unfortunately, the various curves in [15] are mutually incompatible, which is possibly due to some errors: thus the cause of their loops cannot be ascertained. The loops may not be real phenomena, but created by the apparatus, a suspicion which appears to be substantiated by another paper by Pinnel *et al.* [16] on similar precision measurements.

The effect of residual stress on composite properties is occasionally mentioned in literature, but sometimes the statements are not quite correct.

Mehan [17] takes into account thermal stress more carefully than we have done in Equation 17. He comes to the misleading conclusion that the Al matrix of his composite yields as a result of thermal stress in the vicinity of the fibres. His formula, however, shows that the matrix will yield throughout the composite.

Stuhrke [7] points out a reduction of the elastic limit by residual stress. However, his observation that plastic flow removes the residual stress may be misinterpreted especially as no mention is made about useful residual stresses.

A special instance of the effect of residual stress on $\sigma_{0.2}$ is treated correctly in a monograph by Ivanova *et al.* [18]. Their schematized $\sigma(\epsilon)$ plot, however, shows some features which do not correspond to a real situation.

If residual stress is to be utilized as a means for increasing the composite yield stress, σ_{y1}, as indicated in Section 3.1, it is important to know to what extent residual stress decays with time. Investigations along these lines are very useful and some results have been reported by Parrat [19]. Our investigations on the decay of residual stress are currently being carried out.

Acknowledgement

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